Mini-course title: Systems of parameters and ideals of König type

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Abstract. A famous and fundamental theorem of König [5] says that in a bipartite graph G the minimal size of a vertex cover of G coincides with the induced matching number of G. Nowadays, any graph with this property is called a König graph. In combinatorics there is a vast literature about König graphs and also on generalizations of the König property. In these lectures we study the algebraic significance of the König property. For the edge ideal $I(G)$ of a graph, the König property of G simply means that the height of $I(G)$ coincides with the monomial grade of $I(G)$. Having this in mind, leads to the definition of graded ideals of König type. Monomial ideals of König type admit very nice systems of parameters which provide a strong tool to check Cohen–Macaulayness. These results are based on Serre‘s theorems given in Lectures 4 and 5. In our lectures we carefully develop the algebraic background and provide the proofs of Serre's theorems. Finally the general theory of ideals of König type is applied to ideals arising in combinatorial contexts. A summary for each lecture is presented.

1. Hilbert series

The multiplicity $e(I, M)$ (sometimes called the degree) of a module M with respect to an ideal of definition I plays a prominent role in our theory of ideals of König type. It is defined via Hilbert functions. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a finitely generated graded R_0-algebra, where R_0 is an Artinian local ring and let M be a finitely generated graded R-module. A typical example is the associated graded ring R of a regular local ring S together with the associated graded module M of an S-module N with respect of an ideal of definition. We study the Hilbert function and the Hilbert series of M. We recover the fact that the Hilbert function is of polynomial type and state the Hilbert-Serre theorem on the Hilbert series of M. Moreover, we show that the multiplicity of M is obtained by evaluating the numerator polynomial of Hilbert series of M at 1.

2. Systems of parameters

Let R be a Noetherian local ring or a finitely generated graded K-algebra, where K is a field and let M be a finitely generated (graded) R-module. We apply Krull's generalized principal ideal theorem to introduce systems of parameters and prove their existence which in the graded case can be chosen to be composed by homogeneous elements. Special systems of parameters, naturally appear in the study of ideals of König type. They respect the multigraded structure of the algebras under consideration. We show that a sequence $x_1, \ldots, x_d$ in the (graded) maximal ideal of R is a system of parameters for the d-dimensional module M if and only if $\dim M/(x_1, ..., x_i)M = \dim M - i$ for all i.

3. The multiplicity symbol and the Euler characteristic of Koszul homology

Let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence of elements in the (graded) maximal ideal of R which we assume to be graded in the graded case. Such a sequence is called a multiplicity system of M, if the length $\ell(M/(\mathbf{x})M)$ of $M/(\mathbf{x})M$ is finite. For such a multiplicity system, Northcott introduced the multiplicity symbol $e(\mathbf{x}, M)$ which is defined recursively. The highlight of this lecture is the proof of the Auslander–Buchsbaum theorem which says that $e(\mathbf{x}, M)$ is nothing but the Euler characteristic $\chi(\mathbf{x}, M)$ of the Koszul homology $H(\mathbf{x}; M)$. This result plays a role in Serre's theorem in Lecture 5.

4. Serre's Theorem on the multiplicity symbol for systems of parameters

In this lecture we present the proof of Serre's theorem which relates the Euler characteristic $\chi(\mathbf{x}, M)$ of a multiplicity system $\mathbf{x}$ of the module M to the multiplicity $e(I, M)$ with $I = (\mathbf{x})$. Here the distinguished properties of systems of parameters become apparent. Indeed, Serre's theorem says that $\chi(\mathbf{x}, M) = e(I, M)$ if $\mathbf{x}$ is a system of parameters of M, and otherwise $\chi(\mathbf{x}, M) = 0$. As an application we prove the useful associativity formula for multiplicities. In an important special case the formula says that if the module M has a positive rank, then $e(I, M) = e(I, R)\operatorname{rank} M$.

5. Serre's Theorem on partial Euler characteristics

Let $(R, \mathfrak{m})$ be a Noetherian ring. For a multiplicity system $\mathbf{x}$, the partial Euler characteristics with respect to M are defined to be the numbers $\chi_j(\mathbf{x}, M) = \sum_{i \geq j}(-1)^{i-j}\,\ell(H_i(\mathbf{x}, M))$. In the case that R contains its residue class field, we provide a simple proof of Serre's theorem which asserts that $\chi_j(\mathbf{x}, M) \geq 0$ for all j, and prove $\chi_1(\mathbf{x}, M) \geq 0$ in general. In the case that $\mathbf{x}$ is a parameter system of M, the vanishing of $\chi_1(\mathbf{x}, M)$ is equivalent to saying that $\mathbf{x}$ is an M-sequence or that M is Cohen–Macaulay.

6. A criterion for Cohen-Macaulayness

We use Serre's theorems to deduce a Cohen-Macaulay criteria in terms of parameter systems. Let R be a Noetherian local ring (or a standard graded K-algebra) with (graded) maximal ideal $\mathfrak{m}$. Based on results explained in Lectures 4 and 5, we prove a numerical condition for when a system of parameters for R is a regular sequence. Indeed, let $f_1, \ldots, f_d \in \mathfrak{m}$ be a system of parameters for R, $\overline{R} = R/(f_1, \ldots, f_d)$ and $e(M)$ be the multiplicity of an R-module M. Then $e(\overline{R}) \geq e(R)$, and if $e(\overline{R}) = e(R)$ then $f_1, \ldots, f_d$ is a regular sequence (equivalently, R is Cohen–Macaulay). Moreover, if $(f_1, \ldots, f_d)$ is a reduction ideal of $\mathfrak{m}$ and $f_1, \ldots, f_d$ is a regular sequence, then $e(\overline{R}) = e(R)$. The graded version of this criterion will also be proved.

7. Special systems of parameters and graded ideals of König type

For a graded ideal I of a polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K, the Cohen-Macaulay property of the ring S/I may depend on the characteristic of the base field K in general. When I is a monomial or a binomial ideal, we use the criterion for Cohen-Macaulayness given in Lecture 6 to prove that the existence of systems of parameters in a particular form for S/I imply the characteristic independence of Cohen-Macaulayness. Such a system of parameters we call special, which by definition is a system of parameters $s_1, \ldots, s_d$, where each s_j is either of the form x_i or of the form $x_i - x_j$. In order to get graded ideals which admit special systems of parameters, we introduce graded ideals of König type which generalize edge ideals of König graphs. This definition refers and depends on the choice of a monomial order. Attached to a graded ideal of König type I in the polynomial ring S we consider a natural sequence of linear forms of length $d = \dim S/I$ which has the potential to be a special system of parameters for S/I.

8. Monomial ideals and binomial edge ideals of König type

A characterization for monomial ideals of König type in terms of the existence of special systems of parameters for S/I will be given. By using this characterization, one can present a combinatorial description for the Cohen-Macaulay property of such ideals. It turns out that for any graded ideal of König type with respect to $<$, the Cohen–Macaulayness of $\mathrm{in}_<(I)$ does not depend on the base field. For the edge ideals of König graphs a characterization for the Cohen-Macaulay property and some other algebraic invariants are given in terms of numerical data of the graph.

In the second part, we consider the binomial edge ideal J_G of a graph G and give a characterization for binomial edge ideals of König type in terms of subgraphs of G. This allows us to show that when J_G is of König type, S/J_G has a special system of parameters and its Cohen–Macaulay property does not depend on the base field.

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